

# New Constructions of Zero-Correlation Zone Sequences

Yen-Cheng Liu, Ching-Wei Chen, and Yu T. Su

## Abstract

Three new systematic approaches—a transform domain method and two direct (time domain) synthesis methods—for generating zero-correlation zone (ZCZ) sequences are proposed in this paper. The first approach generates sequence families that achieve the upper bounds on the family size and the ZCZ width for a given sequence period. In addition, each generated sequence is a perfect sequence that has an ideal periodic auto-correlation function. The latter two approaches generalize many existing construction methods in the sense that they offer more flexibilities in the choices of the sequence length, the ZCZ width and the constellation size and so are capable of generating sequence families with the same parameters as those by earlier proposals as well as some that are not achievable by previous known methods. For each approach, we provide examples to numerically illustrate the proposed construction procedures.

## Index Terms

Hadamard matrix, Mutually orthogonal complementary set of sequences, Periodic correlation, Up-sampling, Zero-correlation zone (ZCZ) sequence.

Y.-C. Liu and Y. T. Su (correspondence addressee) are with the Institute of Communications Engineering, National Chiao Tung University, Hsinchu, Taiwan (email: joeism@gmail.com; ytsu@nctu.edu.tw). C.-W. Chen is with National Instruments Taiwan Corp., Taipei, Taiwan (email: penguinjazzzy@gmail.com). The material in this paper was presented in part at the IEEE 2009 International Symposium on Information Theory.

## I. INTRODUCTION

Families of sequences with some desired periodic or aperiodic autocorrelation (AC) and cross-correlation (CC) properties are useful in communication and radar systems for applications in identification, synchronization, ranging, or/and interference mitigation. For example, to minimize the multiple access interference (MAI) and self-interference (e.g., inter-symbol interference) in a multi-user, multi-path environment or to avoid inter-antenna interference in a multiple-input, multiple-output system, one would like to have an *ideal sequence set* whose periodic AC functions are nonzero only at the zeroth correlation lag ( $\tau = 0$ ) and whose pairwise periodic CC values are identically zero at any  $\tau$  for all pairs of sequences. Similar aperiodic properties are called for in designing pulse compressed radar signal or two-dimensional array waveforms to have an impulse-like ambiguity function satisfying the resolution requirements.

Unfortunately, the ideal sequence set does not exist, i.e., it is impossible to have impulse-like AC functions and zero CC functions simultaneously in a sequence set. In fact, bounds on the magnitude of CC and AC values derived in [1] and [2] suggest that the design of sequence sets involves the tradeoff between AC and CC values. An alternate compromise is to require that the ideal AC and CC properties be maintained only at correlation lags within a window called zero-correlation zone (ZCZ). Sequences with such properties are known as ZCZ sequences. Little or no system performance degradation results if the correlation values outside the ZCZ are immaterial to the application of concern. For example, if the maximum channel delay spread  $T_m$  and the maximum distance between a base station and co-channel users  $D_m$  are known, a CDMA system using a family of ZCZ sequences with ZCZ width  $|\tau| \leq T_m + 2D_m/c$ , where  $c$  is the speed of light, will be able to suppress MAI and multipath interference.

Other than the restrictions on the magnitude of correlation values, practical concerns also prefer that the choice of the sequence period be flexible and the family size be as large as possible while keeping the desired AC and CC properties intact. It is known, however, given a sequence period there is a tradeoff between the ZCZ width and the family size [3], [4].

Various ZCZ sequence generation methods have been proposed. [5] and [6] presented methods based on complementary sets. Interleaving techniques are shown to be effective in constructing

ZCZ sequences [7], [8]. Sets of ZCZ sequences derived by manipulating perfect sequences were suggested in [9] and [10]. Park *et al.* [11] construct sequences that has nonzero AC only at subperiodic correlation lags and zero CC functions cross all lags.

In this paper, we present three systematic approaches for generating families of sequences whose periodic AC and CC functions satisfy a variety of ZCZ requirements. While some known ZCZ sequence construction methods involve Hadamard matrices in time domain (e.g., [8], [10]), our first approach uses such matrices to meet the desired transform domain properties of a ZCZ sequence set. Sequence sets generated from this approach are, by construction, optimal in the sense that the upper bounds for family sizes and ZCZ widths are achieved. We further employ a modulation operation which converts sequences of nonconstant modulus symbols into polyphase ones without changing the correlation properties.

Based upon a basic binary sequence (to be defined in Section IV) whose AC function satisfies the ZCZ requirement, the second approach generates ZCZ sequence families by a special nonuniform upsampling based on unitary matrices. The construction of basic sequences seems trivial and straightforward, but from these intuitively simple sequences we are able to synthesize desired polyphase ZCZ sequences through some refining steps that involve nonuniform upsampling and modulation.

Our third approach involves the notion of complementary set of sequences [13], [14]. It bears the flavor of the second approach and makes use of a basic binary sequence which meets the ZCZ constraint as well as a collection of mutually orthogonal complementary sets. While this method is capable of generating binary sequences with sequence parameters identical to those given in [5] and [6], it can also produce sequence sets that are unobtainable by the conventional complementary set-based approaches.

The rest of this paper is organized as follows. We introduce basic definitions and properties related to our investigation in the next section. Section III contain the analysis and synthesis of the proposed transform domain approach. We then show some ZCZ sequence sets generated by the transform domain method in subsection III-D. The direct synthesis method is presented in section IV and construction examples are given in subsection IV-E. In section V, a complemen-

tary sequence set based extension of the second approach is proposed, followed by numerical construction examples given in subsection V-D. Finally, some concluding remarks are provided in Section VI.

## II. DEFINITIONS AND FUNDAMENTAL PROPERTIES

*Definition 1:* We refer to  $\mathbf{X}$  as an  $(N, K)$  sequence set if  $\mathbf{X}$  is a set of  $K$  sequences of period  $N$ . That is, for every sequence  $\{u(n)\} \in \mathbf{X}$ ,  $u(i) = u(i + N)$ ,  $\forall i \in \mathbb{Z}$ , where  $\mathbb{Z}$  is the set of all integers.

*Definition 2:* The periodic CC function of two period- $N$  sequences  $u \equiv \{u(n)\}$  and  $v \equiv \{v(n)\}$  is defined as

$$\theta_{uv}(\tau) = \sum_{n=0}^{N-1} u(n)v^*(n - \tau) = u(\tau) \circledast v^*(-\tau), \quad (1)$$

where  $\circledast$  denotes the circular convolution.

Thus, the periodic AC function of sequence  $u$  is simply  $\theta_{uu}(\tau)$ . Since these CC and AC functions are also of period  $N$ , to simplify the discussion we shall, throughout this paper, limit the representations and examples of sequences or sequence sets to a single period ( $0 \leq \tau \leq N - 1$ ) unless necessary.

*Definition 3:* A sequence  $\{u(n)\}$  that has an impulse-like (or ideal) AC function, i.e.,  $\theta_{uu}(\tau) = \theta_{uu}(0)\delta(\tau)$ , is called a *perfect sequence*.

*Definition 4:* A sequence  $\{u_v(n)\}$  is said to be obtained from *modulating* the sequence  $u = \{u(n)\}$  by the sequence  $v = \{v(n)\}$  of the same period if

$$u_v(n) \stackrel{\text{def}}{=} u(n) \circ v(n) \stackrel{\text{def}}{=} u(n) \circledast v^*(-n) \equiv \theta_{uv}(n) \quad (2)$$

*Definition 5:* A set of  $K$  period- $N$  sequences  $\mathbf{C} = \{C_0, C_1, \dots, C_{K-1}\}$  is called an  $(N, K, T)$  ZCZ sequence family (or set) if  $\theta_{C_i C_j}(\tau) = 0$  and  $\theta_{C_i C_i}(\tau) = \theta_{C_i C_i}(0)\delta(\tau)$ ,  $|\tau|_N \leq T < N$ ,  $\forall C_i, C_j \in \mathbf{C}$ ,  $i \neq j$ , where  $K = |\mathbf{C}|$  is the cardinality of the family,  $T$  the width of the ZCZ, and  $|k|_N \stackrel{\text{def}}{=} k \bmod N$ .

In [3], it was proved that

*Corollary 1:* Given the sequence period  $N$ , the cardinality  $K$  and the ZCZ width  $T$  are upper-bounded by

$$K(T + 1) \leq N. \quad (3)$$

For  $\pm 1$ -valued binary sequence set, the bound becomes more tight [4]

$$KT \leq \frac{N}{2}, \quad K > 1 \quad (4)$$

This corollary describes the fundamental tradeoff among the sequence period, family size, and ZCZ width. For a fixed  $N$ , increasing the family size must be achieved at the cost of reduced ZCZ width and vice versa. Note that for a set with a single perfect sequence, (3) is automatically satisfied because  $K = 1$  and  $T = N - 1$ .

*Definition 6:* An  $N \times N$  matrix  $\mathbf{U}$  is called a *Hadamard matrix* of order  $N$  if and only if it satisfies two conditions:

- i. Unimodularity: the components of  $\mathbf{U}$  are of the same magnitude  $\sqrt{P}$ ;
- ii. Orthogonality:  $\mathbf{U}\mathbf{U}^H = N\mathbf{P}\mathbf{I}_N$  where  $\mathbf{I}_N$  is the  $N \times N$  identity matrix and  $(\cdot)^H$  denotes the conjugate transpose of the enclosed matrix.

*Definition 7:* A Hadamard matrix  $\mathbf{U}$  is called a Butson Hadamard matrix if its entries are  $m$ th roots of unity for some  $m > 0$  [15].

*Definition 8:* The Matrix

$$\mathbf{F}_M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_M^{-1} & W_M^{-2} & \cdots & W_M^{-(M-1)} \\ 1 & W_M^{-2} & W_M^{-4} & \cdots & W_M^{-2(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_M^{-(M-1)} & W_M^{-2(M-1)} & \cdots & W_M^{-(M-1)^2} \end{bmatrix} \quad (5)$$

is called the  $M$ -discrete Fourier transform ( $M$ -DFT) matrix, where  $W_M^k = e^{j2\pi k/M}$ , and its Hermitian  $\mathbf{F}_M^H = \mathbf{F}_M^{-1}$  is called the  $M$ -inverse DFT ( $M$ -IDFT) matrix. The set of complex  $M$ th

roots of unity,  $\{W_M^k : k = 0, 1, \dots, M-1\}$ , is called the  $M$ -ary phase shift keying ( $M$ -PSK) set and a sequence with elements from the  $M$ -PSK constellation is called an  $M$ -PSK sequence or a polyphase sequence in general.

It is easy to see that all DFT matrices are Butson Hadamard matrices.

*Definition 9:* The  $k$ th Kronecker power of matrix  $\mathbf{U}$ , denoted by  $\otimes^k \mathbf{U}$ , is defined as

$$\otimes^k \mathbf{U} = \underbrace{\mathbf{U} \otimes \mathbf{U} \otimes \dots \otimes \mathbf{U}}_{\mathbf{U} \text{ appears } k \text{ times}}, \quad (6)$$

where  $\otimes$  denotes the Kronecker product.

*Definition 10:* The matrices

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (7)$$

and

$$\mathbf{H}_{2^n} = \otimes^n \mathbf{H}_2 = \begin{bmatrix} \mathbf{H}_{2^{n-1}} & \mathbf{H}_{2^{n-1}} \\ \mathbf{H}_{2^{n-1}} & -\mathbf{H}_{2^{n-1}} \end{bmatrix}, \quad n = 2, 3, \dots, \quad (8)$$

are called *Sylvester Hadamard matrices*.

The following lemma is essential to derive our construction methods in the next section.

*Lemma 1:* [15] The Kronecker (tensor) product of any two Hadamard matrices is a Hadamard matrix.

### III. TRANSFORM DOMAIN CONSTRUCTION METHODS

We first review some transform domain properties of sequences and their correlation functions. New ZCZ sequence construction approaches based on transform domain properties are then proposed.

#### A. Useful Transform Domain Properties

We denote by  $\text{DFT}\{u(n)\}$  the DFT of a periodic sequence  $\{u(n)\}$  and by  $\text{IDFT}\{U(k)\}$  the inverse DFT (IDFT) of a periodic transform domain sequence  $\{U(k)\}$ . We then immediately

have

*Lemma 2:* The DFT of the CC function  $\theta_{uv}(\tau)$  of two period- $N$  sequences,  $\{u(n)\}$  and  $\{v(n)\}$ , is equal to  $U(k)V^*(k)$ , where  $\{U(k)\} = \text{DFT}\{u(n)\}$  and  $\{V(k)\} = \text{DFT}\{v(n)\}$ .

As the AC function of  $\{u(n)\}$  can be expressed as  $\theta_{uu}(n) = u(n) \otimes u^*(-n)$ , its DFT is given by  $\Theta_{uu}(k) = |U(k)|^2$ . Therefore, it is straightforward to show

*Corollary 2:* Sequence  $\{u(n)\}$  is a perfect sequence if and only if  $|U(k)|^2$  is constant for all  $k$ .

Based on the above properties, we can easily prove that

*Lemma 3:* The AC and CC functions of a set of sequences are invariant (up to a scaling factor) to modulation if the modulating sequence  $v$  is a perfect sequence.

As we will see in Section IV-E that this lemma makes the modulation operator very useful in transforming a sequence set into one with entries of the sequences taken from a desired constellation while maintaining the correlation properties.

### B. Basic Constructions

We first note that *Lemma 2* implies

$$\theta_{uv}(\tau) = \sum_{k=0}^{N-1} \Theta_{uv}(k) e^{j\frac{2\pi\tau k}{N}} = \sum_{k=0}^{N-1} U(k)V^*(k) e^{j\frac{2\pi\tau k}{N}} \quad (9)$$

where  $\Theta_{uv}(k) = \text{DFT}\{\theta_{uv}(\tau)\}$ . Using the identity

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{m-1} = 0, \quad \forall \alpha = W_m^p, \quad (10)$$

where  $|p|_m \neq 0$ , we have

*Lemma 4:* The CC function  $\theta_{uv}(\tau)$  of two period- $N$  sequences  $\{u(n)\}$  and  $\{v(n)\}$  is identical zero  $\forall |\tau|_N \leq T$  if the associated DFT vectors  $\{U(k)\}$  and  $\{V(k)\}$  are orthogonal and their Hadamard product (with one being conjugated) is periodic with a period  $K = N/(T+1)$ , where  $T \in \mathbb{Z}$ .

The recursive Kronecker construction of the Sylvester Hadamard matrices (8) gives at least two sets of row vectors (i.e., upper- and lower-half parts of  $\mathbf{H}_{2^n}$ ) that satisfy both the orthogonality

and periodicity requirements. This property still holds when we replace Sylvester Hadamard matrices by other classes of Hadamard matrices produced by a recursive Kronecker construction similar to (8). Furthermore, as elements of a Hadamard matrix have constant modulus, the AC of all sequences derived by taking IDFT on rows of a Hadamard matrix is 0 for all nonzero correlation lags by *Lemma 1*. These two observations suggest that ZCZ families can be obtained by using proper subsets of row vectors from a Hadamard matrix. To have a precise definition of “proper subsets,” we need

*Definition 11:* A regular  $p$ th-order  $M$ -partition on an  $N \times N$  matrix  $\mathbf{H}$ , where  $N = M^n$ , is the set of  $m = N/K = M^p$   $K \times N$  submatrices, each is formed by non-overlapping  $K = M^{n-p}$  consecutive rows of  $\mathbf{H}$ .

Proper subsets of row vectors that generate ZCZ families are obtained by performing  $p$ th-order  $M$ -partition on the  $n$ th Kronecker power of a Hadamard matrix, i.e.,

*Lemma 5:* Let  $\mathbf{U}$  be a Hadamard matrix of order  $M$  and  $\mathbf{H}$  be the Hadamard matrix of order  $N$  generated by the  $n$ th Kronecker power of  $\mathbf{U}$ , i.e.,

$$\mathbf{H} = [\mathbf{h}_0^T, \mathbf{h}_1^T, \dots, \mathbf{h}_{N-1}^T]^T = \otimes^n \mathbf{U}, \quad (11)$$

where  $N = M^n$ ,  $n \geq 2$ , and  $\mathbf{h}_\ell$  is the  $\ell$ th row<sup>1</sup> of  $\mathbf{H}$ . Perform a regular  $p$ th-order  $M$ -partition on  $\mathbf{H}$  to obtain  $m = M^p$  submatrices

$$\tilde{\mathbf{H}}_i = [\mathbf{h}_{iK}^T, \dots, \mathbf{h}_{(i+1)K-1}^T]^T, \quad i = 0, 1, \dots, m-1. \quad (12)$$

Then, for each  $i$ , the set of  $K$  length- $N$  sequences  $\mathbf{A}_i \stackrel{\text{def}}{=} \{A_{i,0}, A_{i,1}, \dots, A_{i,K-1}\}$ , where  $A_{i,j} = \text{IDFT}\{\mathbf{h}_{iK+j}\}$ , is an  $(N, K, m-1)$  ZCZ sequence family that achieves the upper bound (3). Furthermore, all member sequences in the family are perfect sequences.

*Proof:* The matrix  $\mathbf{H}$  can be expressed in the stacked form,  $\mathbf{H} = [\tilde{\mathbf{H}}_0^T, \tilde{\mathbf{H}}_1^T, \dots, \tilde{\mathbf{H}}_{m-1}^T]^T$ , where the submatrix  $\tilde{\mathbf{H}}_i$  is of the form

$$[a_{i,0}\mathbf{B}, a_{i,1}\mathbf{B}, \dots, a_{i,m-1}\mathbf{B}]$$

<sup>1</sup>For convenience, all the column, row, and vector elements' indices start with 0 instead of 1.



where  $a_{i,j}$  have unit magnitudes and  $\mathbf{B} = \otimes^{n-p} \mathbf{U}$ . It follows immediately that the Hadamard products of two distinct rows of  $\tilde{\mathbf{H}}_i$  has a period of  $M^{n-p} = K$ . ■

The above construction gives ZCZ sequences of length  $M^n, n \geq 2$ . That the upper bound (3) is achieved is a result of our partition method described by *Definition 11*. The sequence length constraint can be relaxed by using Kronecker construction of Hadamard matrices of different orders. Using *Lemma 1* and an argument similar to that used in *Lemma 5*, we obtain

*Theorem 1:* Let  $\mathbf{H}$  be the  $N \times N$  Hadamard matrix

$$\mathbf{H} = [\mathbf{h}_0^T, \mathbf{h}_1^T, \dots, \mathbf{h}_{N-1}^T]^T \stackrel{\text{def}}{=} \mathbf{U}_{n-1} \otimes \dots \otimes \mathbf{U}_0 \quad (13)$$

where  $\mathbf{U}_k, k = 0, 1, \dots, n-1$ , are  $M_k \times M_k$  (not necessarily distinct) Hadamard matrices,  $N = \prod_{k=0}^{n-1} M_k, n \geq 2$ ,  $\mathbf{h}_\ell$  is the  $\ell$ th row of  $\mathbf{H}$ . Partition  $\mathbf{H}$  into  $m = \frac{N}{K}$   $K \times N$  submatrices,

$$\tilde{\mathbf{H}}_i = [\mathbf{h}_{iK}^T, \dots, \mathbf{h}_{(i+1)K-1}^T]^T, \quad i = 0, 1, \dots, m-1, \quad (14)$$

each formed by non-overlapping  $K = \prod_{k=0}^{n-p-1} M_k$  consecutive rows of  $\mathbf{H}$  with  $p > 0$ . Then, for each  $i$ , the set of  $K$  period- $N$  sequences  $\mathbf{A}_i \stackrel{\text{def}}{=} \{A_{i,0}, A_{i,1}, \dots, A_{i,K-1}\}$ , where  $A_{i,j} = \text{IDFT}\{\mathbf{h}_{iK+j}\}$ , is an  $(N, K, m-1)$  ZCZ sequence family that achieves the upper bound (3)<sup>2</sup>.

Note that the recursive generation of Hadamard matrices defined by (8) and (11) are simply special cases of (13), i.e., the above *theorem* generalize *Theorems 1* and *2* of [16].

### C. Polyphase ZCZ Sequences

The ZCZ sequences generated by the methods described above are not necessary of constant modulus but can be converted into polyphase sequences without altering the desired AC and CC properties by a proper modulation process; see *Definition 4* and *Lemma 3*. To find the modulating perfect sequences we need the following two properties.

*Lemma 6:* [17] Let  $U$  be a length- $N$  polyphase perfect sequence with entries drawn from the  $N$ -PSK constellation. Then both  $\text{IDFT}\{U\}$  and  $\text{DFT}\{U\}$  are polyphase perfect sequences.

<sup>2</sup>Technically, the theorem is also valid for  $p = 0$ , as the resulting set has a ZCZ width 0. We will implicitly ignore this trivial case and assume  $p > 0$  in the subsequent discussion.

*Lemma 7:* [18] Let  $L$  be a natural number and  $N = L^2$ . Define the length- $N$  polyphase sequence  $\{u(k)\}$  by

$$u(k_1L + k_2) = W_L^{\beta(k_2)k_1 + r(k_2)}, \quad 0 \leq k_1, k_2 < L, \quad (15)$$

where  $\{\beta(k_2) : k_2 = 0, 1, \dots, L-1\}$  is a permutation of  $\{0, 1, \dots, L-1\}$ , and  $r(k_2)$  is a rational number depending on  $k_2$ . Then the sequences,  $\{u(k)\}$ ,

$$\{e^{j\theta_{k_2}} u(k_1L + k_2) : 0 \leq \theta_{k_2} < 2\pi, \quad 0 \leq k_1, k_2 < L\} \quad (16)$$

and

$$\{W_L^{\ell k_1} u(k_1L + k_2) : 0 \leq k_1, k_2 < L\}, \text{ for any integer } \ell, \quad (17)$$

are all polyphase perfect sequences.

Based on the above results, we suggest a transform domain construction of polyphase ZCZ sequences as follows.

*Lemma 8:* Let  $\mathbf{u}$  be a length- $N$  perfect sequence of the form (15),  $N = \prod_{k=0}^{n-1} M_k = L^2$  for some  $L$  and  $\tilde{\mathbf{H}}_i$  be the  $i$ th submatrix defined by (14) and (13) using  $M_k$ -DFT or  $M_k$ -IDFT matrices. Then  $\{C_{i,n} \stackrel{\text{def}}{=} \text{IDFT}\{\mathbf{h}_{iK+n}\} \circ \text{IDFT}\{\mathbf{u}\}, 0 \leq n \leq K-1\}$  form an  $(N, K, \frac{N}{K} - 1)$  bound-achieving polyphase ZCZ sequence set.

*Proof:* Since the entries in the  $n$ th row of  $\tilde{\mathbf{H}}_i$  render the general expression

$$[\mathbf{H}]_{iK+n, k_1L+k_2} \stackrel{\text{def}}{=} h_{iK+n}(k_1L + k_2) = e^{j\theta_{k_2}(n)} W_L^{\ell(n)k_1}$$

for  $0 \leq k_1, k_2 < L$ , where  $\ell(n) \in \mathbb{Z}$  and  $0 \leq \theta_{k_2}(n) < 2\pi$ , the products  $h_{iK+n}(k)u^*(k)$  are of the forms (15)–(17) and are integer powers of  $W_N$ . *Lemmas 6 and 7* imply that the sequence

$$C_{i,n}(k) \stackrel{\text{def}}{=} \text{IDFT}\{h_{iK+n}(k)\} \circ \text{IDFT}\{u(k)\} = \text{IDFT}\{h_{iK+n}(k)u^*(k)\}$$

has polyphase entries. Invoking *Theorem 1* and *Lemma 3*, we conclude that  $\{C_{i,n} : 0 \leq n < K\}$  is an  $(N, K, \frac{N}{K} - 1)$  polyphase ZCZ family. ■



$$\begin{aligned}
& W_6^0 W_6^4 W_6^2 W_6^0 W_6^4 W_6^2 W_6^0 W_6^5 W_6^4 W_6^3 W_6^2 W_6^1 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0), \\
C_1 = A_1 \circ U_{36} = & (W_6^0 W_6^5 W_6^4 W_6^3 W_6^2 W_6^1 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^1 W_6^2 W_6^3 W_6^4 W_6^5 \\
& W_6^0 W_6^2 W_6^4 W_6^0 W_6^2 W_6^4 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^4 W_6^2 W_6^0 W_6^4 W_6^2), \\
C_2 = A_2 \circ U_{36} = & (W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^4 W_6^2 W_6^0 W_6^4 W_6^2 W_6^0 W_6^5 W_6^4 W_6^3 W_6^2 W_6^1 \\
& W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^1 W_6^2 W_6^3 W_6^4 W_6^5 W_6^0 W_6^2 W_6^4 W_6^0 W_6^2 W_6^4).
\end{aligned}$$

2) *Construction based on Kronecker power of a DFT matrix:*

*Example 2:* Let  $\mathbf{H} = \mathbf{F}_3 \otimes \mathbf{F}_3 \otimes \mathbf{F}_3 \otimes \mathbf{F}_3$  and denote by  $\tilde{\mathbf{H}}_0, \tilde{\mathbf{H}}_1, \dots, \tilde{\mathbf{H}}_{26}$  the submatrices obtained by performing regular 3rd-order 3-partition on  $\mathbf{H}$ . Choosing  $\tilde{\mathbf{H}}_2$  and performing IDFT on its rows, we obtain sequences  $\{A_0, A_1, A_2\}$ . Modulating them by the following polyphase perfect sequence:

$$\begin{aligned}
U_{81} = & (W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^0 W_9^8 W_9^7 W_9^6 W_9^5 W_9^4 W_9^3 W_9^2 W_9^1 \\
& W_9^0 W_9^7 W_9^5 W_9^3 W_9^1 W_9^8 W_9^6 W_9^4 W_9^2 W_9^0 W_9^6 W_9^3 W_9^0 W_9^6 W_9^3 \\
& W_9^0 W_9^5 W_9^1 W_9^6 W_9^2 W_9^7 W_9^3 W_9^8 W_9^4 W_9^0 W_9^4 W_9^8 W_9^3 W_9^7 W_9^2 W_9^6 W_9^1 W_9^5 \\
& W_9^0 W_9^3 W_9^6 W_9^0 W_9^3 W_9^6 W_9^0 W_9^3 W_9^6 W_9^0 W_9^2 W_9^4 W_9^6 W_9^8 W_9^1 W_9^3 W_9^5 W_9^7 \\
& W_9^0 W_9^1 W_9^2 W_9^3 W_9^4 W_9^5 W_9^6 W_9^7 W_9^8), \quad (19)
\end{aligned}$$

we obtain

$$\begin{aligned}
C_0 = A_0 \circ U_{81} = & (W_9^0 W_9^1 W_9^2 W_9^6 W_9^7 W_9^8 W_9^3 W_9^4 W_9^5 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 \\
& W_9^0 W_9^3 W_9^6 W_9^3 W_9^6 W_9^0 W_9^6 W_9^0 W_9^3 W_9^0 W_9^4 W_9^8 W_9^6 W_9^1 W_9^5 W_9^3 W_9^7 W_9^2 \\
& W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^6 W_9^3 W_9^3 W_9^0 W_9^6 W_9^6 W_9^3 W_9^0 \\
& W_9^0 W_9^7 W_9^5 W_9^6 W_9^4 W_9^2 W_9^3 W_9^1 W_9^8 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 \\
& W_9^0 W_9^0 W_9^0 W_9^3 W_9^3 W_9^3 W_9^6 W_9^6 W_9^6), \\
C_1 = A_1 \circ U_{81} = & (W_9^0 W_9^7 W_9^5 W_9^6 W_9^4 W_9^2 W_9^3 W_9^1 W_9^8 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7
\end{aligned}$$

$$\begin{aligned}
& W_9^0 W_9^0 W_9^0 W_9^3 W_9^3 W_9^3 W_9^6 W_9^6 W_9^6 W_9^0 W_9^1 W_9^2 W_9^6 W_9^7 W_9^8 W_9^3 W_9^4 W_9^5 \\
& W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^3 W_9^6 W_9^3 W_9^6 W_9^0 W_9^6 W_9^0 W_9^3 \\
& W_9^0 W_9^4 W_9^8 W_9^6 W_9^1 W_9^5 W_9^3 W_9^7 W_9^2 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 \\
& W_9^0 W_9^6 W_9^3 W_9^3 W_9^0 W_9^6 W_9^6 W_9^3 W_9^0), \\
C_2 = A_2 \circ U_{81} = & (W_9^0 W_9^4 W_9^8 W_9^6 W_9^1 W_9^5 W_9^3 W_9^7 W_9^2 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 W_9^0 W_9^5 W_9^1 \\
& W_9^0 W_9^6 W_9^3 W_9^3 W_9^0 W_9^6 W_9^6 W_9^3 W_9^0 W_9^7 W_9^5 W_9^6 W_9^4 W_9^2 W_9^3 W_9^1 W_9^8 \\
& W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^8 W_9^7 W_9^0 W_9^0 W_9^0 W_9^3 W_9^3 W_9^3 W_9^6 W_9^6 W_9^6 \\
& W_9^0 W_9^1 W_9^2 W_9^6 W_9^7 W_9^8 W_9^3 W_9^4 W_9^5 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 W_9^0 W_9^2 W_9^4 \\
& W_9^0 W_9^3 W_9^6 W_9^3 W_9^6 W_9^0 W_9^6 W_9^0 W_9^3)
\end{aligned}$$

which form an  $(81, 3, 26)$  ZCZ sequence set that satisfies (3).

3) *Quadriphase sequences derived from a Sylvester Hadamard matrix:*

*Example 3:* Partition the Sylvester Hadamard matrix  $\mathbf{H}_{16}$  into four submatrices  $\tilde{\mathbf{H}}_0, \tilde{\mathbf{H}}_1, \tilde{\mathbf{H}}_2, \tilde{\mathbf{H}}_3$  and select the first submatrix,  $\tilde{\mathbf{H}}_0 = [\mathbf{h}_0^T, \mathbf{h}_1^T, \mathbf{h}_2^T, \mathbf{h}_3^T]^T$ . Modulating the IDFT of  $\mathbf{h}_i$  by

$$U_{16} = (W_4^0 W_4^0 W_4^0 W_4^0 W_4^3 W_4^2 W_4^1 W_4^0 W_4^2 W_4^0 W_4^2 W_4^0 W_4^1 W_4^2 W_4^3), \quad (20)$$

for each  $i$ , we obtain

$$\begin{aligned}
C_0 &= (W_4^0 W_4^1 W_4^2 W_4^3 W_4^0 W_4^2 W_4^0 W_4^2 W_4^0 W_4^3 W_4^2 W_4^1 W_4^0 W_4^0 W_4^0 W_4^0), \\
C_1 &= (W_4^0 W_4^3 W_4^2 W_4^1 W_4^0 W_4^0 W_4^0 W_4^0 W_4^1 W_4^2 W_4^3 W_4^0 W_4^2 W_4^0 W_4^2), \\
C_2 &= (W_4^0 W_4^1 W_4^0 W_4^1 W_4^0 W_4^2 W_4^2 W_4^0 W_4^3 W_4^0 W_4^3 W_4^0 W_4^0 W_4^2 W_4^2), \\
C_3 &= (W_4^0 W_4^3 W_4^0 W_4^3 W_4^0 W_4^2 W_4^2 W_4^0 W_4^1 W_4^0 W_4^1 W_4^0 W_4^2 W_4^2 W_4^0), \quad (21)
\end{aligned}$$

a quadriphase  $(16, 4, 3)$  ZCZ sequence family that satisfies (3).

Note that if a 3rd-order 2-partition is used instead, we have a set of only two sequences but with a larger ZCZ width, i.e., we obtain a quadriphase  $(16, 2, 7)$  ZCZ sequence set consisting

of  $\{A_0 \circ U_{16}, A_1 \circ U_{16}\}$  or  $\{A_2 \circ U_{16}, A_3 \circ U_{16}\}$ .

#### IV. DIRECT SYNTHESIS METHOD

##### A. Preliminaries

We now present an alternate approach which is capable of generating ZCZ sequences of arbitrary non-prime periods.

*Definition 12:* A binary sequence, consists only of 0's and 1's, of period  $N$  which satisfies the ZCZ width constraint  $T$  on its AC function is called a basic  $(N, T)$  sequence.

A basic sequence can be obtained by the simple rule given in

*Lemma 9:* A binary sequence  $B = (b_0, b_1, \dots, b_{N-1})$ ,  $b_i \in \{0, 1\}$  is a basic  $(N, T)$  sequence if the minimum run length of 0's between two consecutive 1's is  $T$  (in the circular sense), where a run refers to a string of identical symbols and  $T$  is also called the *minimum spacing* of  $B$ .

##### B. Synthesis Process

Two new operations are needed.

*Definition 13:* A basic  $(N, T)$  sequence  $B$  with Hamming weight  $w_H(B)$  can be expressed as the sum (via component-wise addition<sup>3</sup>) of  $M$  length- $N$  binary sequences,  $\{B_i\}_{i=1}^M$ , where  $\sum_{i=0}^{M-1} w_H(B_i) = w_H(B)$  and  $w_H(B_i) \geq 1$ . We say  $\{B_i\}_{i=1}^M$  is an *orthogonal tone decomposition* of  $B$ .

It is trivial to see that  $\{B_i\}_{i=1}^M$  is a binary  $(N, M, T)$  ZCZ sequence family.

*Definition 14:* Let  $V = (v(0), v(1), \dots, v(N-1))$  be a length- $N$  binary sequence with Hamming weight  $w_H(V) = k$ ,  $s_V(m) =$  the  $m$ th nonzero coordinate of the sequence  $V$  and  $\mathbf{U} = [u_{ij}]$  be a  $k' \times k$  matrix. The  $V$ -upsampled matrix of  $\mathbf{U}$  is the  $k' \times N$  matrix  $\mathbf{P} = [p_{ij}]$  defined by

$$p_{ij} = \begin{cases} u_{im}, & j = s_V(m), m = 0, 1, \dots, k-1; \\ 0, & \text{otherwise.} \end{cases} \quad (22)$$

<sup>3</sup>Note that the "addition" refers to the ordinary addition rather than the modulo 2 addition. Thus, any two of the  $M$  sequences in the decomposition do not have 1's at the same coordinates.

We denote the above row-wise nonuniform upsampling operation on  $\mathbf{U}$  by  $\mathbf{P} = \mathbf{U} \triangle V$ .

Obviously, the nonzero entries in all rows of the matrix  $\mathbf{P} = \mathbf{U} \triangle V$  are in the same positions. Hence if  $V$  is an  $(N, T)$  basic sequence constructed by following the procedure described in *Lemma 9*, then each row has the same minimum spacing  $T$  and all CC (including AC) values are zero at  $0 < \tau \leq T$ . Values of all CC functions at  $\tau = 0$  are zero when  $\mathbf{U}$  is unitary in which case rows of  $\mathbf{P}$  all have ZCZ width  $T$ . Invoking *Lemma 3*, we have

*Lemma 10:* Let  $B$  be a basic  $(N, T)$  sequence with  $w_H(B) = K$  and  $\mathbf{B} \stackrel{\text{def}}{=} \{B_i\}_{i=0}^{M-1}$  be an orthogonal tone decomposition of  $B$ . Then the rows of  $M$  nonuniform upsampled matrices  $\mathbf{P}_i = \mathbf{U}_i \triangle B_i, i = 0, \dots, M-1$ , where  $\mathbf{U}_i$  are  $K_i \times K_i$  (not necessarily distinct) unitary matrices with  $K_i = w_H(B_i)$ , form an  $(N, w_H(B), T)$  ZCZ sequence family. Furthermore, modulating each row of  $\mathbf{P}_i$ 's by a perfect sequence of length  $N$ , we obtain another  $(N, w_H(B), T)$  ZCZ sequence family.

### C. Polyphase ZCZ Sequences

The above process does not guarantee a constant modulus constellation for the entries of the generated sequences. We need a special class of basic sequences and a suitable perfect sequence to generates polyphase sequence families.

*Theorem 2:* Let  $A' = \{a'_n\}$  be a length- $N'$  perfect  $N_{A'}$ -PSK sequence, where  $2 \leq N_{A'} \leq 2N'$  and  $A$  be the perfect sequence of length  $N = N_r N'$  derived from  $N_r$ -fold upsampling on  $A'$ . An  $(N, N_r, N' - 1)$  or  $(N, N_r, N' - 2)$  ZCZ  $\zeta$ -phase sequence family, where  $\zeta = \text{lcm}(N_{A'}, N_r)$ , can be obtained by following the synthesis procedure of *Lemma 10* with the perfect sequence  $A$  and  $\mathbf{P} = \mathbf{F}_{N_r} \triangle B$ , where  $B$  is the weight- $N_r$  basic sequence  $B = (b_0, b_1, \dots, b_{N-1})$  defined by

$$b_i = \begin{cases} 1, & i = kN', \quad k = 0, 1, \dots, N_r - 1; \\ 0, & \text{otherwise,} \end{cases} \quad (23)$$

if  $N_r$  and  $N'$  are relatively prime, or by

$$b_i = \begin{cases} 1, & i = kN', \quad k = 0, 1, \dots, \frac{L_0}{N'} - 1, \text{ or} \\ & i = \ell L_0 + \left(\frac{N}{L_0} - \ell\right) + kN', \text{ where} \\ & \ell = 1, 2, \dots, \frac{N}{L_0} - 1, \\ & k = 0, 1, \dots, \frac{L_0}{N'} - 1; \\ 0, & \text{otherwise,} \end{cases} \quad (24)$$

if  $\gcd(N_r, N') \neq 1$ , where  $L_0 = \text{lcm}(N_r, N')$ .

*Proof:* See Appendix B. ■

#### D. Properties, constraints, and comparisons

The following three properties about the approach described above are easily verifiable.

- (P1) For a fixed  $N$  and  $K = N_r$ , ZCZ sequence families generated by (23) achieve the upper bound (3) and those generated from (24) satisfy the relation  $K(T + 1) = N - N_r$ .
- (P2) The basic sequences defined by (23) and (24) can be cyclically shifted to generate distinct polyphase ZCZ sequence families without affecting the properties mentioned above.
- (P3) To generate binary ZCZ sequences one can use binary Hadamard matrices, which exist for  $N_r = 2^\ell$ ,  $12 \times 2^\ell$ , or  $20 \times 2^\ell$  [15], to replace the  $N_r$ -DFT matrix,  $\mathbf{F}_{N_r}$ , in constructing  $\mathbf{P}$  and reduce the required constellation size to just  $\text{lcm}(N_{A'}, 2) = 2$ ; see *Examples 8 and 9* in the next subsection.

A few remarks on the parameter selection constraints and related comparisons with some existing methods are summarized below.

- (R1) *Theorem 2* does not explicitly mention any restriction on the constellation size. As these constructions need to use a length- $N'$  perfect sequence and  $N_r \times N_r$  Hadamard matrices, which do not always exist for all length ( $N'$ ) and all constellation sizes ( $N_{A'}, N_r$ ), the ZCZ width, sequence length and family size are thus implicitly constrained by the constellation size.
- (R2) Tang *et al.* [7] classifies the ZCZ sequences construction methods into two major categories,



i.e., i) those based on complementary sets and ii) those derived from perfect sequences. The direct synthesis approach belongs to the latter category and generates sequences with lengths  $N = n_1 n_2$ , where  $n_1$  is the length of a perfect sequences. The constructions proposed in [4] and [7]–[10] have similar constraints on the sequence length  $N$  in addition to those mentioned in the next three remarks.

- (R3) In [4], an  $(N, k, (n_1 - 2)k^{\ell-1})$  set is constructed by using a length- $n_1$  perfect sequence but  $n_2$  is restricted to  $k^\ell$ ,  $\ell > 1$ . The interleaving scheme [7] requires that either (i)  $\gcd(n_1, n_2) = 1$  or (ii)  $n_1 | n_2$  or  $n_2 | n_1$  to generate an  $(N, n_2, n_1 - 1)$  or  $(N, n_2, n_1 - 2)$  ZCZ family. The length constraints in (i) is similar to that for the construction (23) while (ii) leads to ZCZ families of the same parameters as those by the construction (24) except that the latter is only constrained by  $\gcd(n_1, n_2) \neq 1$ .
- (R4) In [9], the choice of the perfect sequence length  $n_1$  is limited to  $(2k+1)(2t+1)$ , where  $k \geq 0$  and  $t \geq 1$ , and the resulting family has the parameters  $(4(2k+1)(2t+1), 2(2k+1), 4t+1)$ .
- (R5) A length- $N$  ( $N = n_1 n_2$ ) Frank-Chu perfect sequence is used in [10] to generate an  $(N, n_2, n_1 - 1)$  family. This method also calls for the use of an  $n_2 \times n_2$  DFT or binary Hadamard matrix. However, for the case when  $n_1$  is a perfect square and a DFT (or binary Hadamard) matrix is used, we need a constellation of size  $\text{lcm}(n_2, \sqrt{n_1})$  or  $\text{lcm}(2, \sqrt{n_1})$  instead of  $\text{lcm}(n_2, n_1)$ ,  $\text{lcm}(n_2, 2n_1)$  or  $\text{lcm}(2, n_1)$  required by [10]. Moreover, as [10] is primarily interested in polyphase (nonbinary) sequences, their approach is not applicable for binary set since it requires  $n_1 = 2$ . Our constructions, on the other hand, can be applied to generate both binary and nonbinary families.
- (R6) The construction based on (23) generates sequences that possess the same correlation properties as those of the so-called PS sequences [11]. These sequences are bound-achieving; they have nonzero AC values only on subperiodic correlation lags at  $\tau = n(T+1)$ ,  $n \in \mathbb{Z}$ , and zero CC across all lags. While the PS sequences require that  $\gcd(n_1, n_2) = 1$ , where  $n_1$  is a perfect square, to construct an  $(N, n_2, n_1 - 1)$  family, our method does not impose any constraint on  $n_1$ . Moreover, when  $n_1$  is a perfect square, our approach can generate sequences, which, for the convenience of reference, is called *PS-like sequences*, that requires

a constellation of size  $\text{lcm}(n_2, \sqrt{n_1}) = N/\sqrt{n_1}$  as opposed to  $\text{lcm}(n_1, n_2) = N$  required by the PS approach [11]. Similarly, we refer to those families derived from (23) using non-perfect square  $n_1$  as *generalized PS sequences* for these sequences cannot be generated by the PS method. Some PS-like and generalized PS sequence sets are given in the following subsection.

#### E. Examples of Direct Synthesized Sequence Sets

*Example 4:* (PS-Like sequences) Following the procedure described in *Theorem 2* with  $N_r = 2$ ,  $N' = 9$ , using  $B = (100000000100000000)$  and Sylvester Hadamard matrix  $\mathbf{H}_2$ , we obtain  $\mathbf{P} = \mathbf{U} \triangle B = [P_0^T, P_1^T]^T$  with

$$\begin{aligned} P_0 &= (1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), \\ P_1 &= (1, 0, 0, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 0). \end{aligned} \quad (25)$$

Modulating them by the upsampled perfect sequence  $A = (W_3^0 0 W_3^0 0 W_3^0 0 W_3^2 0 W_3^1 0 W_3^0 0 W_3^1 0 W_3^2 0)$ , we have

$$\begin{aligned} C_0 &= P_0 \circ A = (W_6^0 W_6^2 W_6^2 W_6^0 W_6^4 W_6^0 W_6^0 W_6^0 W_6^4 W_6^0 W_6^2 W_6^2 W_6^0 W_6^4 W_6^0 W_6^0 W_6^4), \\ C_1 &= P_1 \circ A = (W_6^3 W_6^2 W_6^5 W_6^0 W_6^1 W_6^0 W_6^3 W_6^0 W_6^1 W_6^0 W_6^5 W_6^2 W_6^3 W_6^4 W_6^3 W_6^0 W_6^3 W_6^4). \end{aligned} \quad (26)$$

It can be shown that

$$\theta_{C_0 C_1}(\tau) = 0, \quad |\theta_{C_0 C_0}(\tau)| = |\theta_{C_1 C_1}(\tau)| = 18\delta(|\tau|_9).$$

This set of PS-like sequences, (26), is an  $(18, 2, 9)$  bound-achieving ZCZ sequence family.

*Example 5:* (Length-12 PS-like sequences) A set of three PS-like sequences can be generated with  $N_r = 3$ ,  $N' = 4$ ,  $B = (100010001000)$ ,

$$\mathbf{U} = \begin{bmatrix} W_3^0 & W_3^0 & W_3^0 \\ W_3^0 & W_3^1 & W_3^2 \\ W_3^0 & W_3^2 & W_3^1 \end{bmatrix} \quad (27)$$

$A = (1, 0, 0, 1, 0, 0, 1, 0, 0, -1, 0, 0)$  and

$$\begin{aligned} P_0 &= (W_3^0 000 W_3^0 000 W_3^0 000), \\ P_1 &= (W_3^0 000 W_3^1 000 W_3^2 000), \\ P_2 &= (W_3^0 000 W_3^2 000 W_3^1 000). \end{aligned} \quad (28)$$

The resulting ZCZ sequences are:

$$\begin{aligned} C_0 &= P_0 \circ A = (W_6^0 W_6^0 W_6^0 W_6^3 W_6^0 W_6^0 W_6^0 W_6^3 W_6^0 W_6^0 W_6^0 W_6^3), \\ C_1 &= P_1 \circ A = (W_6^0 W_6^2 W_6^4 W_6^3 W_6^2 W_6^4 W_6^0 W_6^5 W_6^4 W_6^0 W_6^2 W_6^1), \\ C_2 &= P_2 \circ A = (W_6^0 W_6^4 W_6^2 W_6^3 W_6^4 W_6^2 W_6^0 W_6^1 W_6^2 W_6^0 W_6^4 W_6^5). \end{aligned} \quad (29)$$

It is verifiable that  $\forall i, j, i \neq j$ ,

$$\theta_{C_i C_j}(\tau) = 0, \quad |\theta_{C_i C_i}(\tau)| = 12\delta(|\tau|_4), \quad (30)$$

i.e.,  $\mathbf{C} = \{C_0, C_1, C_2\}$  is a  $(12, 3, 3)$  bound-achieving ZCZ sequence set. This set also possesses the same PS sequence correlation properties [11]. Moreover, both (26) and (29) require only  $1/3$  and  $1/2$  of the constellation sizes required by the original PS sequences under the same sequence period constraints.

*Example 6:* (Generalized PS sequences) Following the method of *Theorem 2* with  $N_r = 5$ ,  $N' = 3$ ,  $B = (100100100100100)$ , the DFT matrix

$$\mathbf{U} = \begin{bmatrix} W_5^0 & W_5^0 & W_5^0 & W_5^0 & W_5^0 \\ W_5^0 & W_5^1 & W_5^2 & W_5^3 & W_5^4 \\ W_5^0 & W_5^2 & W_5^4 & W_5^1 & W_5^3 \\ W_5^0 & W_5^3 & W_5^1 & W_5^4 & W_5^2 \\ W_5^0 & W_5^4 & W_5^3 & W_5^2 & W_5^1 \end{bmatrix}, \quad (31)$$

and  $A = (W_3^0 0000 W_3^2 0000 W_3^0 0000)$ , we obtain

$$\begin{aligned}
C_0 &= (W_{15}^0 W_{15}^5 W_{15}^0 W_{15}^0 W_{15}^5 W_{15}^0 W_{15}^0 W_{15}^5 W_{15}^0 W_{15}^5 W_{15}^0 W_{15}^0 W_{15}^5 W_{15}^0), \\
C_1 &= (W_{15}^0 W_{15}^{11} W_{15}^{12} W_{15}^3 W_{15}^{14} W_{15}^0 W_{15}^6 W_{15}^2 W_{15}^3 W_{15}^9 W_{15}^5 W_{15}^6 W_{15}^{12} W_{15}^8 W_{15}^9), \\
C_2 &= (W_{15}^0 W_{15}^2 W_{15}^9 W_{15}^6 W_{15}^8 W_{15}^0 W_{15}^{12} W_{15}^{14} W_{15}^6 W_{15}^3 W_{15}^5 W_{15}^{12} W_{15}^9 W_{15}^{11} W_{15}^3), \\
C_3 &= (W_{15}^0 W_{15}^8 W_{15}^6 W_{15}^9 W_{15}^2 W_{15}^0 W_{15}^3 W_{15}^{11} W_{15}^9 W_{15}^{12} W_{15}^5 W_{15}^3 W_{15}^6 W_{15}^{14} W_{15}^{12}), \\
C_4 &= (W_{15}^0 W_{15}^{14} W_{15}^3 W_{15}^{12} W_{15}^{11} W_{15}^0 W_{15}^9 W_{15}^8 W_{15}^{12} W_{15}^6 W_{15}^5 W_{15}^9 W_{15}^3 W_{15}^2 W_{15}^6) \quad (32)
\end{aligned}$$

which constitute a set of  $(15, 5, 2)$  bound-achieving generalized PS sequences that has the same correlation properties as the original PS sequences, i.e.,  $\forall i, j, i \neq j$ ,

$$\theta_{C_i C_j}(\tau) = 0, \quad |\theta_{C_i C_i}(\tau)| = 15\delta(|\tau|_3). \quad (33)$$

As mentioned before, the PS method [11] can not produce ZCZ sequences of length  $N = 15$ . Previous examples are constructed by using coprime  $N_r$  and  $N'$ , we present a set using the construction (24).

*Example 7:* (More generalized PS sequences) By choosing  $N_r = 4$ ,  $N' = 6$  and upsampling the Sylvester Hadamard  $\mathbf{H}_4$  by  $B = (10000010000001000001 \ 0000)$ , we obtain  $\mathbf{P} = \mathbf{H}_4 \triangle B$ . A  $(24, 4, 4)$  ZCZ sequence family is then generated by modulating each row of the resulting matrix  $\mathbf{P}$  by  $A = (W_{12}^0 000 W_{12}^1 000 W_{12}^4 000 W_{12}^9 000 W_{12}^4 \ 000 W_{12}^1 000)$ :

$$\begin{aligned}
C_0 &= P_0 \circ A = (W_{12}^0 W_{12}^3 W_{12}^{11} W_{12}^8 W_{12}^{11} W_{12}^8 W_{12}^0 W_{12}^3 W_{12}^8 W_{12}^{11} W_{12}^{11} W_{12}^8 \\
&\quad W_{12}^3 W_{12}^0 W_{12}^8 W_{12}^{11} W_{12}^8 W_{12}^{11} W_{12}^3 W_{12}^0 W_{12}^{11} W_{12}^8 W_{12}^8 W_{12}^{11}), \\
C_1 &= P_1 \circ A = (W_{12}^0 W_{12}^3 W_{12}^5 W_{12}^2 W_{12}^{11} W_{12}^8 W_{12}^6 W_{12}^9 W_{12}^8 W_{12}^{11} W_{12}^5 W_{12}^2 \\
&\quad W_{12}^3 W_{12}^0 W_{12}^2 W_{12}^5 W_{12}^8 W_{12}^{11} W_{12}^9 W_{12}^6 W_{12}^{11} W_{12}^8 W_{12}^2 W_{12}^5), \\
C_2 &= P_2 \circ A = (W_{12}^0 W_{12}^9 W_{12}^{11} W_{12}^2 W_{12}^{11} W_{12}^2 W_{12}^0 W_{12}^9 W_{12}^8 W_{12}^5 W_{12}^{11} W_{12}^2 \\
&\quad W_{12}^3 W_{12}^6 W_{12}^8 W_{12}^5 W_{12}^8 W_{12}^5 W_{12}^3 W_{12}^6 W_{12}^{11} W_{12}^2 W_{12}^8 W_{12}^5), \\
C_3 &= P_3 \circ A = (W_{12}^0 W_{12}^9 W_{12}^5 W_{12}^8 W_{12}^{11} W_{12}^2 W_{12}^6 W_{12}^3 W_{12}^8 W_{12}^5 W_{12}^5 W_{12}^8)
\end{aligned}$$

$$W_{12}^3 W_{12}^6 W_{12}^2 W_{12}^{11} W_{12}^8 W_{12}^5 W_{12}^9 W_{12}^0 W_{12}^{11} W_{12}^2 W_{12}^2 W_{12}^{11}).$$

We can also derive a smaller *generalized PS sequence* set with a larger ZCZ width by using  $N' = 8$ ,  $N_r = 3$ , and (23). Specifically,

$$\begin{aligned} C_0 &= (W_{12}^9 W_{12}^3 W_{12}^6 W_{12}^3 W_{12}^9 W_{12}^9 W_{12}^6 W_{12}^9 W_{12}^9 W_{12}^3 W_{12}^6 W_{12}^3 \\ &\quad W_{12}^9 W_{12}^9 W_{12}^6 W_{12}^9 W_{12}^9 W_{12}^3 W_{12}^6 W_{12}^3 W_{12}^9 W_{12}^9 W_{12}^6 W_{12}^9), \\ C_1 &= (W_{12}^9 W_{12}^7 W_{12}^2 W_{12}^3 W_{12}^1 W_{12}^5 W_{12}^6 W_{12}^1 W_{12}^5 W_{12}^3 W_{12}^{10} W_{12}^{11} \\ &\quad W_{12}^9 W_{12}^1 W_{12}^2 W_{12}^9 W_{12}^1 W_{12}^{11} W_{12}^6 W_{12}^7 W_{12}^5 W_{12}^9 W_{12}^{10} W_{12}^5), \\ C_2 &= (W_{12}^9 W_{12}^{11} W_{12}^{10} W_{12}^3 W_{12}^5 W_{12}^1 W_{12}^6 W_{12}^5 W_{12}^1 W_{12}^3 W_{12}^2 W_{12}^7 \\ &\quad W_{12}^9 W_{12}^5 W_{12}^{10} W_{12}^9 W_{12}^5 W_{12}^7 W_{12}^6 W_{12}^{11} W_{12}^1 W_{12}^9 W_{12}^2 W_{12}^1). \end{aligned}$$

These three sequences form an  $(N, N_r, N' - 1) = (24, 3, 7)$  bound-achieving ZCZ sequence set. A family with such ZCZ parameter values can not be generated by the method suggested in [11] where  $N'$  must be a perfect square.

*Example 8:* (Length-16 ternary and binary sequences) Using the basic sequence  $B = (1000000100100100)$ , the Sylvester Hadamard matrix  $\mathbf{H}_4$ , and the perfect sequence  $A = (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, -1, 0, 0, 0)$ , we obtain

$$\begin{aligned} P_0 &= (+, 0, 0, 0, 0, 0, 0, +, 0, 0, +, 0, 0, +, 0, 0), \\ P_1 &= (+, 0, 0, 0, 0, 0, 0, -, 0, 0, +, 0, 0, -, 0, 0), \\ P_2 &= (+, 0, 0, 0, 0, 0, 0, +, 0, 0, -, 0, 0, -, 0, 0), \\ P_3 &= (+, 0, 0, 0, 0, 0, 0, -, 0, 0, -, 0, 0, +, 0, 0), \end{aligned} \tag{34}$$

where  $+$  and  $-$  denote  $+1$  and  $-1$ , respectively. Time domain sequences with zero entries are often undesirable as they involve on-off switching. Modulating  $\{P_i\}$  by  $A$ , we obtain a binary

(16, 4, 2) ZCZ sequence family consisting of

$$C_0 = P_0 \circ A = (+ - + + - + + + + + - + + - +),$$

$$C_1 = P_1 \circ A = (+ + + - - - + - + - + + + - - -),$$

$$C_2 = P_2 \circ A = (+ + - + - - - + + - - - + - + +),$$

$$C_3 = P_3 \circ A = (+ - - - - + - - + + - + + + + -).$$

*Example 9: (Length-32 binary sequences)* Let  $B = (10001000000100010010001001000100)$ ,  $N_r = 8$ ,  $N' = 4$  and  $A = (+ 0000000 + 0000000 + 0000000 - 0000000)$ . With  $\mathbf{H}_8$ , we obtain

$$C_0 = (+ - + + + - + + - + + + - + + + + + - + + + - + + - + + + - +),$$

$$C_1 = (+ - + + - + - - - + + + + - - - + + + - - - - + + + - + - - + -),$$

$$C_2 = (+ + + - + + + - - - + - - - + - + - + + + - + + + - - - + - - -),$$

$$C_3 = (+ + + - - - - + - - + - + + - + + - + + - + - - + - - - - + + +),$$

$$C_4 = (+ + - + + + - + - - - + - - - + + - - - + - - - + - + + + - + +),$$

$$C_5 = (+ + - + - - + - - - - + + + + - + - - - - + + + + - + + - + - -),$$

$$C_6 = (+ - - - + - - - - + - - - + - - + + - + + + - + + + + - + + + -),$$

$$C_7 = (+ - - - - + + + - + - - + - + + + + - + - - + - + + + - - - - +),$$

a binary (32, 8, 2) ZCZ sequence set.

The ZCZ families shown in the above two examples achieve (4), the bound for binary ( $N_{A'} = 2$ ) sequences, but their ZCZ widths are limited by the facts that there exists only one binary perfect sequence (whose length  $N' = 4$ ) and binary Hadamard matrices only exists for certain  $N_r$ ; see (R3) and (R4). To increase the ZCZ width and have greater flexibility in choosing the ZCZ parameters, we can use higher-order constellations ( $N_r > 2$ ). For example, quadriphase perfect sequences of length  $N' = 2, 4, 8$  or  $16$  do exists [18], [19]. We introduce in the next

section an alternate method which can extend the ZCZ width of a binary ZCZ sequence set while maintaining the family cardinality.

## V. SEQUENCES DERIVED FROM COMPLEMENTARY SETS OF SEQUENCES

In this section, we generalize the above basic sequence based approach by replacing rows of an unitary matrix with concatenated sequences. The following definitions can be found in [14].

### A. Basic Definitions

*Definition 15:* The aperiodic CC function of two length- $L$  sequences  $u \equiv \{u(n)\}$  and  $v \equiv \{v(n)\}$  is defined as

$$\psi_{uv}(\tau) = \sum_{n=\tau}^{L-1} u(n)v^*(n-\tau). \quad (35)$$

The aperiodic AC function of sequence  $u$  is obviously  $\psi_{uu}(\tau)$ .

*Definition 16:* A set of  $Q$  equal-length sequences,  $\mathbf{E} = \{E_0, E_1, \dots, E_{Q-1}\}$ , forms a *complementary set* of sequences (CSS) if and only if  $\forall \tau \neq 0$ ,

$$\sum_{i=0}^{Q-1} \psi_{E_i E_i}(\tau) = 0. \quad (36)$$

*Definition 17:* A CSS,  $\mathbf{F} = \{F_0, F_1, \dots, F_{Q-1}\}$ , is said to be a *mate* of the CSS,  $\mathbf{E} = \{E_0, E_1, \dots, E_{Q-1}\}$  if

- i. The lengths of all members in  $\mathbf{E}$  and  $\mathbf{F}$  are the same;
- ii. For all  $\tau$ ,

$$\sum_{i=0}^{Q-1} \psi_{E_i F_i}(\tau) = 0. \quad (37)$$

*Definition 18:* A collection of complementary sets of sequences  $\{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$ , where each set contains the same number of sequences, is said to be *mutually orthogonal* if every two sets in the collection are mates of each other.

Note that it has been shown in [13] that

*Corollary 3:* The number of mutually orthogonal CSS's (MOCSS's)  $K$  cannot exceed that of sequences in a set  $Q$ , i.e.,  $K \leq Q$ .

### B. Synthesis Procedure

We now extend the nonuniform upsampling operation defined in *Definition 14*.

*Definition 19:* Let  $V$  be a length- $N$  binary sequence with  $w_H(V) = Q$  and  $\mathcal{E} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$  be a collection of  $K$  MOCSS's in which each CSS  $\mathbf{E}_i$  consists of  $Q$  length- $L$  sequences, i.e.,  $\mathbf{E}_i = \{E_{i0}, E_{i1}, \dots, E_{i(Q-1)}\}$ , where  $E_{ij} = (e_{ij}(0), e_{ij}(1), \dots, e_{ij}(L-1))$ .

The  $V$ -upsampled concatenated sequence based on  $\mathbf{E}_i$ ,  $G_i = \mathbf{E}_i \triangle_c V = (g_i(0), g_i(1), \dots, g_i(N + Q(L-1) - 1))$  is defined by

$$g_i(n) = \begin{cases} e_{ij}(m), & n = j(L-1) + s_V(j) + m, \\ 0, & \text{otherwise.} \end{cases} \quad (38)$$

The operator  $\triangle_c$  is similar to  $\triangle$  which operates on rows of a matrix while the former operates on the sequence formed by concatenating members of the set  $\mathbf{E}_i$ . It replaces each nonzero element of a basic sequence by a finite-length sequence.

*Lemma 11:* Let  $\mathcal{E} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$  be a collection of  $K$  MOCSS's in which each set  $\mathbf{E}_i$  has  $Q$  length- $L$  sequences and  $B$  be a basic  $(N, T)$  sequence of weight  $Q$ . The set  $\mathbf{C} = \{(\mathbf{E}_i \triangle_c B) \circ A\} \stackrel{\text{def}}{=} \{G_i \circ A\} \stackrel{\text{def}}{=} \{C_0, C_1, \dots, C_{K-1}\}$  with  $A$  being a perfect sequence of length  $N + Q(L-1)$  forms an  $(N + Q(L-1), K, T)$  ZCZ sequence family.

*Proof:* Based on *Lemma 9* and *Definition 19* we can express  $G_i$  as

$$G_i = \underbrace{(0 \cdots 0)}_{s_V(0)} \underbrace{e_{i0}(0) \cdots e_{i0}(L-1)}_L \underbrace{0 \cdots 0}_{s_V(1) - s_V(0) - 1} \\ \underbrace{e_{i1}(0) \cdots e_{i1}(L-1)}_L \underbrace{0 \cdots 0}_{s_V(2) - s_V(1) - 1} \\ \vdots \\ \underbrace{e_{i(Q-1)}(0) \cdots e_{i(Q-1)}(L-1)}_L \underbrace{0 \cdots 0}_{N - s_V(Q-1) - 1},$$



where  $s_V(j) - s_V(j-1) - 1 \geq T$ ,  $j = 0, 1, \dots, Q-1$ , and  $s_V(0) - s_V(Q-1) + N - 1 \geq T$ . Invoking *Definitions 17* and *18*, we obtain, for all  $i \neq k$ ,

$$\theta_{G_i G_k}(\tau) = \sum_{j=0}^{Q-1} \psi_{E_{ij} E_{kj}}(\tau) = 0, \quad |\tau|_N \leq T. \quad (39)$$

By analogy, *Definition 16* gives, for all  $i$ ,  $\theta_{G_i G_i}(\tau) = \sum_{j=0}^{Q-1} \psi_{E_{ij} E_{ij}}(\tau) = 0$ ,  $0 < |\tau|_N \leq T$ . Therefore,  $\mathbf{G}$  forms an  $(N + Q(L-1), K, T)$  ZCZ sequence set.  $\mathbf{C}$  is also an  $(N + Q(L-1), K, T)$  ZCZ sequence set as *Lemma 3* ensures that the correlation properties after modulating all  $G_i$ 's by  $A$  remain unchanged. ■

### C. Polyphase ZCZ Sequences

Following the idea described in Section IV-C, we can derive another class of polyphase ZCZ sequence families by using suitable perfect sequences and basic sequences. The proof of the next corollary is similar to that of *Theorem 2* and is given in the last two paragraphs of Appendix A.

*Corollary 4:* Let  $A$  be the length- $LN$  perfect sequence obtained by  $LN_r$ -fold upsampling on a length- $N'$  perfect  $N_{A'}$ -phase sequence,  $A'$ , where  $N = N_r N'$  and  $2 \leq N_{A'} \leq N'$ . Denote by  $\mathcal{E} = \{\mathbf{E}_0, \mathbf{E}_1, \dots, \mathbf{E}_{K-1}\}$  a collection of  $K$  MOCSS', where  $K \leq N_r$  and each CSS  $\mathbf{E}_i = \{E_{i0}, E_{i1}, \dots, E_{i(N_r-1)}\}$  contains  $N_r$  length- $L$   $N_c$ -phase sequences. An  $(LN, K, T)$  ZCZ  $\zeta$ -phase sequence set,  $\zeta = \text{lcm}(N_{A'}, N_c)$ , with  $T = L(N' - 2)$  if  $\text{gcd}(N_r, N') \neq 1$  or  $T = L(N' - 1)$  if  $\text{gcd}(N_r, N') = 1$  can be obtained by the following steps:

1. Generate  $K$  length- $(N + N_r(L-1))$  sequences  $G'_i = \mathbf{E}_i \triangle_c B$ ,  $i = 0, 1, \dots, K-1$ , where  $B$  is the weight- $N_r$  basic sequence of length  $N$  defined by (24) if  $\text{gcd}(N_r, N') \neq 1$  or by (23) if  $\text{gcd}(N_r, N') = 1$ .
2. Replace each zero in  $G'_i$  by a length- $L$  all-zero sequence to obtain the augmented sequence  $G_i$ .
3. Modulate each  $G_i$  by  $A$ .

We have a few remarks on the MOCSS-based approach.

(R7) Similar to (P2), the basic sequence  $B$  can also be cyclically shifted to generate different polyphase ZCZ families without changing the original correlation properties.

- (R8) As mentioned in Section IV, binary sequence sets constructed by *Theorem 2* have less choices in ZCZ widths. The construction described in *Corollary 4* takes advantage of the fact that the member sequence of an MOCSS exists for many values of  $L$  and thus allow the ZCZ width to be chosen from the set  $\{T = 2L\}$  with the same basic sequence  $B$ , set cardinality  $N_r$ , and perfect sequence  $A'$ .
- (R9) [5] and [6] present MOCSS-based methods for generating binary ZCZ sequences. The ZCZ parameters achieved by these sequences can also be obtained by using our approach described above. For example, a method given in [5] involves a class of recursively generated families of binary CSS  $\{\Delta_n\}$ . Expressing a family of  $Q$  MOCSS's in matrix form [14]

$$\Delta_1 \stackrel{def}{=} \begin{bmatrix} E_{00} & E_{10} & \cdots & E_{(Q-1)0} \\ E_{01} & E_{11} & \cdots & E_{(Q-1)1} \\ \vdots & \vdots & \ddots & \vdots \\ E_{0(Q-1)} & E_{1(Q-1)} & \cdots & E_{(Q-1)(Q-1)} \end{bmatrix}$$

where  $E_{ij}$  are length- $L$  binary sequences and each row is a CSS. Then, for  $n \geq 2$ ,

$$\Delta_n = \begin{bmatrix} \Delta_{n-1} \diamond \Delta_{n-1} & -\Delta_{n-1} \diamond \Delta_{n-1} \\ -\Delta_{n-1} \diamond \Delta_{n-1} & \Delta_{n-1} \diamond \Delta_{n-1} \end{bmatrix},$$

where the  $(i, j)$ th entry of the submatrix  $[\mathbf{A} \diamond \mathbf{B}]$ ,  $[\mathbf{A} \diamond \mathbf{B}]_{ij}$  is obtained by concatenation of the two sequences  $[\mathbf{A}]_{ij}$  and  $[\mathbf{B}]_{ij}$ . The concatenation of rows of  $\Delta_n$  forms a  $(4^{n-1}LQ, 2^{n-1}Q, 2^{n-2}L)$  ZCZ sequence set. On the other hand, by using  $A' = (1, 1, 1, -1)$ , the basic sequence defined by (24) and the family of MOCSS  $\Delta_n$  with  $N_r = 2^{n-1}Q$  and elements of  $\Delta_1$  being length  $L/4$  sequences, we obtain binary ZCZ sequence sets with the parameters  $(4^{n-1}4(L/4)Q, 2^{n-1}Q, 2^{n-1}(L/4) \cdot 2) = (4^{n-1}LQ, 2^{n-1}Q, 2^{n-2}L)$  via *Corollary 4*.

- (R10) Our approach, however, offers more choices in parameter values and thus can produce sets which are not derivable from the methods of [5], [6]. More importantly, we can generate not only binary but also non-binary sequences and the ZCZ parameters for the non-binary



*Example 11:* ( $\gcd(N_r, N') = 1$ ) Using the construction (23) with (40),  $A' = (W_3^0 W_3^2 W_3^0)$ ,  $N' = 3$ ,  $N_r = 4$ , and  $L = 4$ , we can obtain a ZCZ sequence set  $\mathbf{C}$  of the same (or larger)  $T$  with a shorter sequence period  $LN$  and slightly larger constellation:

$$\begin{aligned}
C_0 &= (W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^5 W_6^2 W_6^5 W_6^2 W_6^3 W_6^3 W_6^0 W_6^0 W_6^0 W_6^0 W_6^2 W_6^5 W_6^5 W_6^2 \\
&\quad W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^2 W_6^2 W_6^2 W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^5 W_6^5 W_6^2 W_6^2), \\
C_1 &= (W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^5 W_6^2 W_6^2 W_6^5 W_6^3 W_6^3 W_6^3 W_6^0 W_6^0 W_6^3 W_6^3 W_6^2 W_6^5 W_6^2 W_6^5 \\
&\quad W_6^3 W_6^0 W_6^0 W_6^3 W_6^3 W_6^3 W_6^3 W_6^2 W_6^2 W_6^5 W_6^5 W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^3 W_6^5 W_6^5 W_6^5 W_6^5), \\
C_2 &= (W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^2 W_6^2 W_6^2 W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^0 W_6^5 W_6^5 W_6^2 W_6^2 \\
&\quad W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^5 W_6^2 W_6^5 W_6^2 W_6^3 W_6^3 W_6^0 W_6^0 W_6^0 W_6^0 W_6^0 W_6^2 W_6^5 W_6^5 W_6^2 W_6^2), \\
C_3 &= (W_6^3 W_6^0 W_6^0 W_6^3 W_6^3 W_6^3 W_6^3 W_6^2 W_6^2 W_6^5 W_6^5 W_6^0 W_6^3 W_6^0 W_6^3 W_6^3 W_6^0 W_6^0 W_6^3 W_6^5 W_6^5 W_6^5 W_6^5 \\
&\quad W_6^0 W_6^0 W_6^3 W_6^3 W_6^0 W_6^3 W_6^0 W_6^3 W_6^5 W_6^2 W_6^2 W_6^5 W_6^3 W_6^3 W_6^3 W_6^0 W_6^0 W_6^3 W_6^3 W_6^2 W_6^5 W_6^2 W_6^5).
\end{aligned}$$

It is worth mentioning that the above set cannot be obtained by using the methods of [5] and [6] and, moreover, although *Corollary 4* promises an  $(LN, N_r, L(N' - 1)) = (48, 4, 8)$  family,  $\mathbf{C}$  is actually a  $(48, 4, 9)$  one. The larger ZCZ is due to the inherit correlation properties of MOCSS (40)

$$\sum_{k=0}^{N_r-1} \psi_{E_{ik} E_{j|k \pm 1|_{N_r}}}(\tau) = 0 \quad (41)$$

for  $\tau = \pm(L - 1)$ ,  $0 \leq i < K$ , and  $0 \leq j < K$ .

## VI. CONCLUSION

In this paper, we present three classes of systematic approaches for constructing ZCZ sequence families. The first approach is based on the transform domain ZCZ width requirement and a transform domain AC property. The two other approaches begin with simple binary ZCZ sequences which are progressively refined via Hadamard matrix or CSS-based nonuniform upsampling and perfect sequence-based modulation. The latter operation along with a judicious choice of the perfect sequence are crucial in converting a non-constant modulus sequence into

a polyphase sequence.

We show that our approaches are capable of generating many known binary and polyphase ZCZ families as well as new ones with desired parameters  $(N, K, T)$  which are otherwise unachievable by existing methods. They are also less constrained by the constellation size. Finally, for each approach, numerical examples have been provided to further validate the proposed construction methods.

## APPENDIX A

### PROOFS OF THEOREM 2 AND COROLLARY 4

Let  $P = (p_0, p_1, \dots, p_{N-1})$  be a row of  $\mathbf{P}$  and  $C = P \circ A = (c_0, c_1, \dots, c_{N-1})$ , where  $c_n = \sum_{j=0}^{N-1} p_j a_{[j-n]_N}^*$ . If we can show that, for any  $n \in [0, N-1]$ , one and only one of the  $N$  products  $\{p_j a_{[j-n]_N}^* : j = 0, \dots, N-1\}$  is nonzero, then, as both  $A$  and  $P$  consist 0's and polyphase elements,  $C$  is a polyphase sequence as well. Because of the circular convolution nature of the modulation operation (4) and the periodic run property of  $P$ , we have only to check if this single nonzero product assertion is valid for  $0 < n < N_r$ .

For the first construction (23),  $\gcd(N_r, N') = 1$  and both  $N_r$  and  $N'$  are positive, hence  $\exists$  unique  $a, b \in \mathbb{Z}$  such that  $aN' + bN_r = 1$ , where one of the integer coefficients  $a$  or  $b$  must be negative. Without loss of generality, we assume  $b < 0$  and multiply both sides of the above Bézout's identity by  $s$ ,  $0 < s < N_r$ , to obtain  $saN' = s + sb'N_r$ ,  $b' = -b > 0$ . If  $sb' \leq N' - 1$  then  $saN' < N'N_r = N$  and  $sa < N_r$ ; otherwise, subtract both sides by  $n_0N$ , where  $n_0 = \lfloor \frac{sb'N_r}{N} \rfloor$  to obtain  $(sa - n_0N_r)N' = s + (sb' - n_0N')N_r$ . For both cases, we have, for each positive  $s < N_r$ ,  $\exists$  unique pair of positive integers  $(m, n)$ ,  $0 < m \leq N_r - 1$ ,  $0 \leq n \leq N' - 1$  such that  $mN' = s + nN_r \pmod{N}$ . That this property holds for  $s = 0$  is obvious.

For the second construction (24), we notice that the basic sequence admits the orthogonal tone decomposition,  $B = \sum_{\ell=0}^{d-1} B_\ell$ , where

$$B_\ell(i) = \begin{cases} b_i, & \ell L_0 \leq i < (\ell + 1)L_0; \\ 0, & \text{otherwise.} \end{cases} \quad (42)$$

For  $d = \gcd(N_r, N')$ , there exists positive integers  $a, b'$  such that  $aN' = d + b'N_r$ . Multiplying both sides by  $s$ ,  $0 \leq s < \frac{N}{d}$ , we obtain  $(sa - n_0 \frac{N_r}{d})N' = sd + (sb' - n_0 \frac{N'}{d})N_r$ , where  $n_0 = \left\lfloor \frac{saN'}{L_0} \right\rfloor$ . For all  $s \in \left\{0, 1, \dots, \frac{\lfloor N_r/N' \rfloor N' + (N - N_r)}{d}\right\}$ ,  $\exists$  a unique integer pair  $(m, n)$ ,  $0 \leq m < \frac{N_r}{d}$ ,  $0 \leq n < N'$  such that  $mN' = sd + nN_r \bmod N$ , i.e., the sequence  $B_0 \circ A$  is identically zero except at indices that are multiples of  $d$  and the nonzero terms are the products of two polyphase signals whence are themselves polyphase signals.

Similarly, we can show that, for  $\ell = 1, 2, \dots, d-1$ , the sequence  $B_\ell \circ A$ , has nonzero polyphase terms at  $nd - \ell$  only, where  $n \in \mathbb{Z}$ . Hence the sequence  $B \circ A = \sum_{\ell=0}^{d-1} B_\ell \circ A$  is a polyphase sequence.

To prove *Corollary 4*, we first note that the sequences generated differs from those generated by *Theorem 2* in that the perfect sequence used in *Corollary 4* is the  $L$ -fold upsampled version of that used in *Theorem 2* while the unmodulated ZCZ sequences for the former is an  $L$ -expanded version of those for the latter, replacing each zero entry of  $P$  by a length- $L$  string of zeros and each nonzero entry by a complementary sequence  $E_{ij}$  of length  $L$ .

For the first construction of  $B$  (23), we immediately have, for  $0 \leq s < N_r$ ,  $\exists$  unique pair of positive integers  $(m, n)$ ,  $0 < m \leq N_r - 1$ ,  $0 \leq n \leq N' - 1$  such that  $mLN' = sL + nLN_r = k + nLN_r$ . That is, in computing the modulated sequence  $C = G \circ A = \{c_k\}$ , where  $G = \{g_k\} \stackrel{\text{def}}{=} G_i$  and  $c_k = \sum_{j=0}^{LN-1} g_j a_{|j-k|_{LN}}^*$ , there is only one nonzero term in the summands that add up to  $c_k$ , for  $k = sL$ ,  $s = 0, 1, \dots, N_r - 1$ . That this single nonzero convolution term property holds for  $sL < k < (s+1)L$  is obvious because of the special structure of  $G_i$ . The proof for the case when the second construction (24) is employed follows a similar line of argument.

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